Classical and Quantum Fields in Curved Spacetime:

Canonical Theory versus Conventional Construction

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Abstract

We argue that the conventional construction for quantum fields in curved spacetime has a grave drawback: It involves an uncountable set of physical field systems which are nonequivalent with respect to the Bogolubov transformations, and there is, in general, no canonical way for choosing a single system. Thus the construction does not result in a definite theory. The problem of ambiguity pertains equally both to quantum and classical fields. The canonical theory is advanced, which is based on a canonical, or natural choice of field modes. The principal characteristic feature of the theory is relativistic-gravitational nonlocality: The field at a spacetime point (s, t) depends on the metric at t in the whole 3-space. The most fundamental and shocking result is the following: In the case of a free field in curved spacetime, there is no particle creation. Applications to cosmology and black holes are given. The results for particle energies qqare in complete agreement with those of general relativity. A model of the universe is advanced, which is an extension of the Friedmann universe; it lifts the problem of missing dark matter.

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Introduction

The conventional construction in quantum field theory in curved spacetime is usually presented in such a form [1,2] that the fulfillment of the canonical commutation relations is not evident. That was the reason for a criticism against the construction in our paper [3]. But as it may be seen from a general approach [4] and will be shown in this paper, the commutation relations may be fulfilled. This is the answer to the criticism.

But now we argue that the conventional construction has the following grave drawback: It involves an uncountable set of physical field systems which are nonequivalent with respect to the Bogolubov transformations, and there is, in general, no canonical way for choosing a single system. Thus the conventional construction does not result in a definite field theory. This is the reason for our using the term “construction” rather than “theory”.

The problem of ambiguity is not specifically concerned with quantum fields: for classical ones the situation is the same. Thus both classical and quantum fields should be treated on equal terms.

A field as a dynamical variable (observable) is presented in the form of the expansion in terms of modes, coefficients being the canonical dynamical variables. The problem of fixing the field reduces to that of settling the modes.

In the conventional construction, the basic condition on the field is that it obey the Klein-Gordon equation. Under this condition, there exists a vast arbitrariness for the mode settling. It should be particularly emphasized that the ambiguity involved has nothing to do both with different representations of the same field connected by the Bogolubov transformations and with canonical/unitary transformations connecting equivalent fields.

In the canonical theory, the basic condition is that the mode settling be canonical, or natural, i.e., based on the spacetime structure only. Under this condition, the modes are settled unambiguously, which results in a definite field theory—the canonical one.

Dynamics is presented both in Liouville/Schrödinger and Hamilton/Heisenberg pictures. The quantum Hamiltonian implies the existence of particles, their energies being time dependent. The most fundamental and shocking result is the following: In the case of a free quantum field in curved spacetime, there is no particle creation.

The Klein-Gordon equation is violated in the generic case of a nonstationary metric. A local change in the metric results in changing modes and frequencies/energies. We call this phenomenon relativistic-gravitational nonlocality. Nonlocality is incompatible with the local principle of covariance. The canonical theory meets the geometric principle, which is more general: Spacetime structure and dynamics should be phrased in geometric form.

Applications to cosmology and black holes are given. The results for particle energies are in complete agreement with those of general relativity. A model of the universe is advanced, which is an extension of the Friedmann universe; the model lifts the problem of the missing dark matter.

1 Preliminaries: The algebraic description of a nonautonomous system

A classical/quantum physical system is described in terms of dynamical variables, states, and time evolution, or dynamics.

1.1 Statics: Dynamical variables, states, and mean values

A set $\mathcal{A}$ of dynamical variables A,

$$\{A : A \in \mathcal{A}\}, \tag{1.1.1}$$

and a set Ω of states ω, which are functionals on $\mathcal{A}$,

$$\{\omega : \omega \in \Omega\}\,, \tag{1.1.2}$$

are given,

$$\omega(A) \tag{1.1.3}$$

being the mean value of a dynamical variable A in a state ω.

It is convenient to introduce a quantity ρ which is equivalent to ω,

$$\rho \leftrightarrow \omega, \quad \omega(A) = \langle \rho, A\rangle\,. \tag{1.1.4}$$

1.2 Dynamics: Two pictures

For a nonautonomous system, a dynamical variable is a given function of time,

$$A = A(t) \in \mathcal{A}. \tag{1.2.1}$$

There are two pictures for describing the time dependence of mean values:

$$[\omega(A(t))]_t = \omega_t(A(t)) = \omega(A_t(t)); \tag{1.2.2}$$

for some time t_0,

$$\omega_{t_0} = \omega, \qquad A_{t_0}(t_0) = A(t_0). \tag{1.2.3}$$

The dynamical equations are

$$\frac{d\rho_t}{dt} = -\mathrm{i}[H(t), \rho_t], \tag{1.2.4}$$

$$\frac{dA_t(t)}{dt} = \left(\frac{dA(t)}{dt}\right)_t + \mathrm{i}[H_t(t), A_t(t)], \tag{1.2.5}$$

where H is the Hamiltonian.

1.3 A classical system

A phase space Γ is given,

$$\{\gamma : \gamma \in \Gamma\}, \qquad \gamma = \Big\{(\alpha_j, \alpha_j^*) : \alpha_j \in C,\ j \in J\Big\}, \tag{1.3.1}$$

Γ being a differentiable manifold. A dynamical variable

$$A = A(\gamma). \tag{1.3.2}$$

ρ is the distribution function,

$$\rho = \rho(\gamma). \tag{1.3.3}$$

The mean value

$$\langle \rho, A \rangle = \int d\gamma \rho(\gamma) A(\gamma), \qquad d\gamma = \prod_j d\alpha_j d\alpha_j^*. \tag{1.3.4}$$

For a pure state,

$$\omega_{\gamma_0} \leftrightarrow \rho_{\gamma_0}(\gamma) = \delta(\gamma - \gamma_0), \qquad \omega_{\gamma_0}(A) = A(\gamma_0). \tag{1.3.5}$$

$\mathrm{i}[A, B]$ is the Poisson bracket,

$$[A, B] = \sum_j \left(\frac{\partial A}{\partial \alpha_j} \frac{\partial B}{\partial \alpha_j^*} - \frac{\partial A}{\partial \alpha_j^*} \frac{\partial B}{\partial \alpha_j} \right) = \frac{1}{\mathrm{i}} \sum_j \left(\frac{\partial A}{\partial p_j} \frac{\partial B}{\partial q_j} - \frac{\partial A}{\partial q_j} \frac{\partial B}{\partial p_j} \right), \tag{1.3.6}$$

where

$$\alpha_j = \frac{1}{\sqrt{2}} \left(\sqrt{\omega_j} q_j + \frac{\mathrm{i}}{\sqrt{\omega_j}} p_j \right), \qquad \alpha_j^* = \frac{1}{\sqrt{2}} \left(\sqrt{\omega_j} q_j - \frac{\mathrm{i}}{\sqrt{\omega_j}} p_j \right), \tag{1.3.7}$$

ω_j is arbitrary and in the general case has nothing to do with the Hamiltonian.

In the Liouville picture,

$$A(t) = A_L(\gamma, t) \equiv A(\gamma, t). \tag{1.3.8}$$

For the sake of brevity, we denote the canonical dynamical variables by

$$\alpha_j(\gamma) \equiv A(\gamma) \quad \text{where} \quad A(\gamma) = \alpha_j, \qquad \alpha_j^*(\gamma) = A^*(\gamma); \tag{1.3.9}$$

$\alpha_j(\gamma),\ \alpha_j^*(\gamma)$ are time independent. We have

$$[\alpha_j, \alpha_k] = 0,\ [\alpha_j^*, \alpha_k^*] = 0,\ [\alpha_j, \alpha_k^*] = \delta_{jk}. \tag{1.3.10}$$

In the Hamilton picture,

$$A_t(t) = A_{Ht}(\gamma, t) \equiv A_t(\gamma, t) = A(\gamma_t, t). \tag{1.3.11}$$

The equations

$$\frac{d\alpha_{jt}}{dt} = -\mathrm{i} \frac{\partial H(\gamma_t, t)}{\partial \alpha_{jt}^*}, \qquad \frac{d\alpha_{jt}^*}{dt} = \mathrm{i} \frac{\partial H(\gamma_t, t)}{\partial \alpha_{jt}} \tag{1.3.12}$$

generate

$$\gamma_t = \mathcal{G}_{t,t_0} \gamma; \tag{1.3.13}$$

$\mathcal{G}_{t,t_0}\gamma$ gives for a fixed γ a curve and for a fixed t a transformation $\Gamma \to \Gamma$; $\mathcal{G}_{t,t_0}^{-1}$ is the inverse transformation.

We have

$$\int d\gamma \rho_t(\gamma) A(\gamma, t) = \int d\gamma \rho(\gamma) A_t(\gamma, t), \tag{1.3.14}$$

$$\rho(\gamma) \equiv \rho_H(\gamma), \qquad \rho_t(\gamma) \equiv \rho_{Lt}(\gamma) = \rho(\mathcal{G}_{t,t_0}^{-1} \gamma), \tag{1.3.15}$$

$$A_t(\gamma, t) = A(\mathcal{G}_{t,t_0} \gamma, t). \tag{1.3.16}$$

1.4 A quantum system

A separable Hilbert space $\mathcal{H}$ is given. A dynamical variable A is an operator, ρ is the statistical operator,

$$\langle \rho, A \rangle = \mathrm{Tr}\{\rho A\}. \tag{1.4.1}$$

In the Schrödinger picture,

$$A_S(t) \equiv A(t), \qquad \rho_{St} \equiv \rho_t. \tag{1.4.2}$$

In the Heisenberg picture,

$$A_{Ht}(t) \equiv A_t(t), \qquad \rho_H \equiv \rho. \tag{1.4.3}$$

We have

$$\rho_t = U_{t,t_0} \rho U_{t_0,t}, \qquad A_t(t) = U_{t_0,t} A(t) U_{t,t_0}, \tag{1.4.4}$$

$$U_{t,t_0} = T \exp\left\{ -\mathrm{i} \int\limits_{t_0}^{t} H(t')dt' \right\}, \qquad H(t) \equiv H_S(t). \tag{1.4.5}$$

The Hilbert space $\mathcal{H}$ may be realized as the Fock space constructed on the annihilation and creation operators $a_j,\ a_j^\dagger$,

$$[a_j, a_k] = 0, \qquad [a_j^\dagger, a_k^\dagger] = 0, \qquad [a_j, a_k^\dagger] = \delta_{jk}; \tag{1.4.6}$$

then

$$A(t) = A(\nu, t), \qquad \nu = \left\{ (a_j, a_j^\dagger) : j \in J \right\}. \tag{1.4.7}$$

1.5 Classical-quantum relation

The classical-quantum relation is given by

$$\alpha_j \leftrightarrow a_j, \quad \alpha_j^* \leftrightarrow a_j^\dagger, \tag{1.5.1}$$

$$A(\gamma, t) \leftrightarrow A(\nu, t). \tag{1.5.2}$$

2 Field

2.1 Field, momentum, and Hamiltonian

In a comoving reference frame, metric is of the form

$$g = g(x,t) = (dt)^2 - h_{ik}(x,t) dx^i dx^k, \tag{2.1.1}$$

and the Hamiltonian in the Liouville/Schrödinger picture is

$$H(t) = \frac{1}{2} \int_S dx \sqrt{|h(x,t)|} \left\{ \pi^2(x,t) + h^{ik}(x,t) \partial_i \phi(x,t) \partial_k \phi(x,t) + m^2 \phi^2(x,t) \right\} \tag{2.1.2}$$

where the field $\phi(x,t)$ and the momentum $\pi(x,t)$ are dynamical variables.

2.2 The problems of phase/Hilbert space and field-momentum

In classical field theory, we should define a phase space Γ and the dynamical variables ϕ, π as functions on it; in quantum field theory, we should fix a Hilbert space $\mathcal{H}$ and the operators ϕ, π in it.

2.3 A straightforward approach

In classical theory, a straightforward approach would be as follows:

$$\gamma = \{(\alpha_s, \alpha_s^*) : s \in S\} \leftrightarrow \{(\phi_s, \pi_s) : s \in S\} \equiv \chi, \tag{2.3.1}$$

$$\phi(\chi, s, t) = \phi_s, \qquad \pi(\chi, s, t) = \pi_s \tag{2.3.2}$$

(cf. (1.3.9)). But there is no canonical, i.e., natural way for introducing manifold structure in the set $\{\chi\}$.

In quantum theory, there is no canonical way for choosing the representation of ϕ, π, the commutation relations only being given.

2.4 Mode approach

In the classical case, we put

$$\gamma = \left\{(\alpha_j, \alpha_j^*) : j \in J\right\}, \qquad \|J\| = \aleph_0 \quad (\text{or } \aleph), \tag{2.4.1}$$

$$\Gamma = l_2 \quad (\text{or } L_2), \tag{2.4.2}$$

$$\phi(\gamma, x, t) = \frac{1}{\sqrt{2}} \sum_j \left\{ \frac{1}{\sqrt{\omega_j(t)}} u_j(x,t)\alpha_j + \frac{1}{\sqrt{\omega_j^*(t)}} u_j^*(x,t)\alpha_j^* \right\} \quad \left(\text{or } \int dj\right), \tag{2.4.3}$$

$$\pi(\gamma, x, t) = \frac{\mathrm{i}}{\sqrt{2}} \sqrt{\frac{|h_u(x,t)|}{|h(x,t)|}} \sum_j \left\{ -\sqrt{\omega_j^*} u_j(x,t)\alpha_j + \sqrt{\omega_j(t)} u_j^*(x,t)\alpha_j^* \right\}, \tag{2.4.4}$$

where

$$u_j^* = u_{p(j)}, \quad \omega_{p(j)} = \omega_j, \tag{2.4.5}$$

p is a permutation, such that

$$p \circ p = \mathrm{I}, \quad p^{-1} = p, \tag{2.4.6}$$

and

$$\int_S dx \sqrt{|h_u(x,t)|}\, u_j^*(x,t) u_k(x,t) = \delta_{jk}, \tag{2.4.7}$$

so that we obtain the canonical commutation relations:

$$\begin{gathered}
[\phi_t(x_1,t), \phi_t(x_2,t)] = 0, \qquad [\pi_t(x_1,t), \pi_t(x_2,2)] = 0, \\
[\phi_t(x_1,t), \pi_t(x_2,t)] = \mathrm{i}\sqrt{\tfrac{|h_u(x_2,t)|}{|h(x_2,t)|}} \textstyle\sum_j u_j(x_1,t) u_j^*(x_2,t) = \mathrm{i}\frac{\delta(x_1 - x_2)}{\sqrt{|h(x_2,t)|}} = \mathrm{i}\delta_{h(s_2,t)}(s_1,s_2), \\
s_1, s_2 \in S.
\end{gathered} \tag{2.4.8}$$

In the quantum case, we substitute ν for γ:

$$\gamma \to \nu. \tag{2.4.9}$$

2.5 The problem of basic elements

The basic elements of the construction being developed are modes u_j's and "frequencies" ω_j's. The problem is in choosing a family of them,

$$\{(u_j, \omega_j) : j \in J\}. \tag{2.5.1}$$

3 The conventional construction

3.1 Basic elements, field and momentum

The conventional construction is realized by the following choice of the basic elements:

$$\frac{\partial u_j}{\partial t} = 0, \quad \frac{\partial \omega_j}{\partial t} = 0, \quad \frac{\partial h_u}{\partial t} = 0, \tag{3.1.1}$$

$$\int_S dx \sqrt{|h_u(x)|} u_j^*(x) u_k(x) = \delta_{jk}, \tag{3.1.2}$$

otherwise the choice is arbitrary.

Thus in the classical case

$$\phi(\gamma, x, t) = \phi(\gamma, x) = \frac{1}{\sqrt{2}} \sum_j \left\{ \frac{1}{\sqrt{\omega_j}} u_j(x) \alpha_j + \frac{1}{\sqrt{\omega_j^*}} u_j^*(x) \alpha_j^* \right\}, \tag{3.1.3}$$

$$\pi(\gamma, x, t) = \pi(\gamma, x) = \frac{\mathrm{i}}{\sqrt{2}} \sqrt{\frac{|h_u(x)|}{|h(x,t)|}} \sum_j \left\{ -\sqrt{\omega_j^*} u_j(x) \alpha_j + \sqrt{\omega_j} u_j^*(x) \alpha_j^* \right\}. \tag{3.1.4}$$

The quantum case is obtained by the substitution (2.4.9).

3.2 Dynamics

We have

$$\frac{\partial \phi}{\partial t} = 0, \quad \frac{\partial(\sqrt{h}\pi)}{\partial t} = 0, \tag{3.2.1}$$

which implies that the Klein-Gordon equation is fulfilled in the Hamilton/Heisenberg picture,

$$(\Box + m^2)\phi_t = 0. \tag{3.2.2}$$

Furthermore,

$$\alpha_{jt} = \sum_l [\xi_{jlt} \alpha_l + \eta_{jlt} \alpha_l^*], \tag{3.2.3}$$

so that

$$\phi_t(\gamma, x) = \sum_l \{ f_{lt} \alpha_l + f_{lt}^* \alpha_l^* \}, \qquad \text{class.}, \tag{3.2.4}$$

$$\phi_t(\nu, x) = \sum_l \{ f_{lt} a_l + f_{lt}^* a_l^* \}, \qquad \text{quant.}, \tag{3.2.5}$$

the functions f_{lt}'s meeting the equations

$$(\Box + m^2) f_{lt} = 0, \tag{3.2.6}$$

$$(f_{lt}, f_{l't})_\Omega = \delta_{ll'} \qquad \text{and so on,} \tag{3.2.7}$$

where $(\cdot, \cdot)_\Omega$ is the symplectic inner product of solutions to the Klein-Gordon equation [1,2,5].

3.3 Particle creation

Let us put

$$h_u = h(t_0), \tag{3.3.1}$$

$$\triangle(t_o)u_j = -k_j^2 u_j, \tag{3.3.2}$$

$$\omega_j = \sqrt{k_j^2 + m^2}. \tag{3.3.3}$$

Then in the quantum case, using the normal ordering we obtain

$$H(t_0) = \sum_j \omega_j a_j^\dagger a_j. \tag{3.3.4}$$

Let

$$\rho = \rho_{t_0} = |\text{vac}\rangle\langle\text{vac}|, \quad a_j|\text{vac}\rangle = 0. \tag{3.3.5}$$

We have

$$H_{t_0}(t_0)|\text{vac}\rangle = H(t_0)|\text{vac}\rangle = 0. \tag{3.3.6}$$

But we obtain for $t \neq t_0$

$$H_t(t)|\text{vac}\rangle \neq 0, \tag{3.3.7}$$

which may be interpreted as particle creation, the result being independent of measurements.

3.4 Ambiguity: Uncountable set of nonequivalent field systems

In view of subsection 3.1, the conventional construction involves an uncountable set of field systems which are nonequivalent with respect to the Bogolubov transformations: The families $\{f_{lt}\}$ relating to different systems are not connected by those transformations. To see this, it suffices to change one frequency in eq.(3.1.3). It should be particularly emphasized that this ambiguity has nothing to do with different representations of the same field $\phi_t(x)$ using the Bogolubov transformations or with canonical/unitary transformations. Even restricting ourselves to the choice given by eqs.(3.3.1)-(3.3.3), in view of the arbitrariness of t_0, we do not eliminate the ambiguity.

Thus the canonical construction does not result in a definite field theory.

4 The canonical theory

The central idea of the canonical theory is as follows. The choice of the basic elements should be canonical, or natural: It should involve only spacetime structure, which is given by spacetime manifold topology and metric.

4.1 Product space-time

The employment of the comoving reference frame implies that spacetime manifold M is a trivial bundle [6], so that we assume from the outset that M is the trivial bundle, i.e., product space-time,

$$M = T \times S, \quad M \ni p = (t, s), \quad t \in T, \quad s \in S, \tag{4.1.1}$$

where T is the cosmic time and S is the cosmic 3-space.

Metric in the comoving reference frame is of the form

$$g = g(s,t) = dt \otimes dt - h(t) = (dt)^2 - h_{ik}(x,t)dx^i dx^k. \tag{4.1.2}$$

4.2 Basic elements, field, momentum, and Hamiltonian

In the Liouville/Schrödinger picture, the Hamiltonian may be presented as

$$H(t) = \frac{1}{2}\int_S dx\sqrt{|h(x,t)|}\left\{\pi^2(x,t) - \phi(x,t)\triangle\phi(x,t) + m^2\phi^2(x,t)\right\}. \tag{4.2.1}$$

A natural, or canonical choice of the basic elements is as follows. We put

$$h_u = h, \tag{4.2.2}$$

$$\triangle(t)u_j = -k_j^2(t)u_j, \quad u_j = u_j(x,t), \tag{4.2.3}$$

$$\omega_j = \omega_j(t) = \sqrt{k_j^2(t) + m^2}. \tag{4.2.4}$$

Now the scalar product is defined by

$$(\varphi_1, \varphi_2)_t = \int_S dx\sqrt{|h(x,t)|}\varphi_1^*(x)\varphi_2(x), \tag{4.2.5}$$

and

$$(u_j, u_k)_t = \delta_{jk}. \tag{4.2.6}$$

We obtain in the classical and quantum cases

$$\phi(\gamma,s,t) = \frac{1}{\sqrt{2}}\sum_j \frac{1}{\sqrt{\omega_j(t)}}\left\{u_j(s,t)\alpha_j + u_j^*(s,t)\alpha_j^*\right\}, \tag{4.2.7}$$

$$\pi(\gamma,s,t) = \frac{\mathrm{i}}{2}\sum_j \sqrt{\omega_j(t)}\left\{-u_j(s,t)\alpha_j + u_j^*(s,t)\alpha_j^*\right\}, \tag{4.2.8}$$

$$H(\gamma,t) = \sum_j \omega_j(t)\alpha_j^*\alpha_j, \tag{4.2.9}$$

and

$$\phi(\nu,s,t) = \frac{1}{\sqrt{2}}\sum_j \frac{1}{\sqrt{\omega_j(t)}}\left\{u_j(s,t)a_j + u_j^*(s,t)a_j^\dagger\right\}, \tag{4.2.10}$$

$$\pi(\nu,s,t) = \frac{\mathrm{i}}{2}\sum_j \sqrt{\omega_j(t)}\left\{-u_j(s,t)a_j + u_j^*(s,t)a_j^\dagger\right\}, \tag{4.2.11}$$

$$H(\nu,t) = \sum_j \omega_j(t)a_j^\dagger a_j \qquad \text{(normal ordering)}, \tag{4.2.12}$$

respectively.

4.3 Dynamics

In the classical case, dynamics is determined by the dynamical variables $\alpha_{jt},\ \alpha^*_{jt}$. We obtain from eqs.(1.3.12),(4.2.9)

$$\alpha_{jt} = \mathrm{e}^{-\mathrm{i}\beta_j(t,t_0)}\alpha_j, \quad \alpha^*_{jt} = \mathrm{e}^{\mathrm{i}\beta_j(t,t_0)}\alpha^*_j, \tag{4.3.1}$$

$$\beta_j(t,t_0) = \int\limits_{t_0}^{t} \omega_j(t')dt'. \tag{4.3.2}$$

Thus the field in the Hamilton picture is

$$\begin{aligned} \phi_t(\gamma,s,t) &= \tfrac{1}{2}\textstyle\sum_j \frac{1}{\sqrt{\omega_j(t)}}\left\{u_j(s,t)\alpha_{jt} + u^*_j(s,t)\alpha^*_{jt}\right\} \\ &= \tfrac{1}{2}\textstyle\sum_j \frac{1}{\sqrt{\omega_j(t)}}\left\{u_{jt}(s,t)\alpha_j + u^*_{jt}(s,t)\alpha^*_j\right\} \end{aligned} \tag{4.3.3}$$

where

$$u_{jt}(s,t) = \mathrm{e}^{-\mathrm{i}\beta_j(t,t_0)}u_j(s,t). \tag{4.3.4}$$

The Hamiltonian

$$H_t(\gamma,t) = H(\gamma,t) = \sum_j \omega_j(t)\alpha^*_j\alpha_j. \tag{4.3.5}$$

In the quantum case, with regard to

$$[H(t_1), H(t_2)] = 0, \tag{4.3.6}$$

we obtain for the time evolution operator (1.4.5)

$$U(t,t_0) = \exp\{-\mathrm{i}\int\limits_{t_0}^{t} H(t')dt'\} = \prod_j \mathrm{e}^{-\mathrm{i}\beta_j(t,t_0)a^\dagger_j a_j}. \tag{4.3.7}$$

We find in the Heisenberg picture

$$a_{jt} = \mathrm{e}^{-\mathrm{i}\beta_j(t,t_0)}a_j, \quad a^\dagger_{jt} = \mathrm{e}^{\mathrm{i}\beta_j(t,t_0)}a^\dagger_j, \tag{4.3.8}$$

and the field operator

$$\phi_t(\nu,s,t) = \frac{1}{\sqrt{2}}\sum_j \frac{1}{\sqrt{\omega_j(t)}}\left\{u_{jt}(s,t)a_j + u^*_{jt}(s,t)a^\dagger_j\right\}. \tag{4.3.9}$$

The Hamiltonian

$$H_t(\nu,t) = H(\nu,t) = \sum_j \omega_j(t)a^\dagger_j a_j. \tag{4.3.10}$$

4.4 Particles

We have

$$H(t) = \sum_j \omega_j(t)N_j, \tag{4.4.1}$$

where in the quantum case

$$N_j = a^\dagger_j a_j \tag{4.4.2}$$

is the occupation number operator. Thus the canonical theory implies the existence of particles,

$$\omega_j(t) = \sqrt{k^2_j(t) + m^2} \tag{4.4.3}$$

being a particle energy.

4.5 No particle creation

We have

$$N_{jt} = N_j, \qquad \frac{dN_{jt}}{dt} = 0. \tag{4.5.1}$$

Thus there is no particle creation.

4.6 The violation of the Klein-Gordon equation

With eqs.(4.3.3),(4.3.9),(4.3.4) in mind, we find

$$(\Box + m^2)\left[\frac{u_{jt}}{\sqrt{\omega_j}}\right] = \frac{1}{\sqrt{|h|}}\frac{\partial\sqrt{|h|}}{\partial t}\frac{\partial}{\partial t}\left[\frac{u_{jt}}{\sqrt{\omega_j}}\right] + \frac{\partial}{\partial t}\left\{\frac{\partial}{\partial t}\left[\frac{u_j}{\sqrt{\omega_j}}\right]\mathrm{e}^{-\mathrm{i}\beta_j}\right\} - \mathrm{i}\frac{\partial}{\partial t}[\sqrt{\omega_j}u_j]\mathrm{e}^{-\mathrm{i}\beta_j}. \tag{4.6.1}$$

Thus the Klein-Gordon equation is violated in the generic case of a nonstationary metric.

The Klein-Gordon equation being abandoned, the equations of motion are those in the Liouville/Schrödinger and Hamilton/Heisenberg pictures.

4.7 Relativistic-gravitational nonlocality

A local change in the metric h results in changing the Laplacian $\triangle$ and, by the same token, solutions to the equation (4.2.3), i.e., k_j^2, u_j, ω_j, and $u_{jt}/\sqrt{\omega_j}$. We call this phenomenon relativistic-gravitational nonlocality.

Generally, relativistic-gravitational nonlocality means that

$$A(\gamma/\nu, s, t) = A[\gamma/\nu, s; h(t)], \tag{4.7.1}$$

i.e., that a Liouville/Schrödinger dynamical variable at a point $p = (s, t)$ depends on the metric $h(t)$ in the whole 3-space S.

The degree of quantum-gravitational nonlocality may be characterized by the quantity

$$b = \frac{\partial}{\partial t}\left[\frac{u}{\sqrt{\omega}}\right]\Big/\left[\frac{u}{\sqrt{\omega}}\right]\omega = \frac{\partial u}{\partial t}\Big/ u\omega - \frac{1}{2}\frac{d\omega}{dt}\Big/\omega^2. \tag{4.7.2}$$

We have

$$\frac{d\omega}{dt} = \frac{k}{\omega}\frac{dk}{dt}, \tag{4.7.3}$$

so that

$$b = \frac{\partial u}{\partial t}\Big/ u\omega - \frac{k}{2\omega^3}\frac{dk}{dt}. \tag{4.7.4}$$

4.8 The geometric principle as an extension of the principle of covariance

Nonlocality is incompatible with the local principle of covariance. More general than the latter is the geometric principle: Spacetime structure and dynamical equations should be phrased in a geometric, coordinate-independent form. The principle of covariance is a local version of the geometric principle.

The canonical theory meets the geometric principle.

4.9 The energy-momentum tensor

For the sake of brevity, from this point on we consider the quantum field. The corresponding results for the classical field are obtained in an obvious way.

Normal ordering on the energy-momentum tensor in the comoving reference frame produces

$$T_{00} = \frac{1}{2} : \left\{ \pi^2 + h^{ik}\partial_i\phi\partial_k\phi + m^2\phi^2 \right\} : , \tag{4.9.1}$$

$$H(t) = \int_S dx \sqrt{|h(t)|} T_{00}, \tag{4.9.2}$$

$$\begin{aligned} T_{ik} =& : \left\{ \partial_i\phi\partial_k\phi + \tfrac{1}{2}h_{ik}[\pi^2 - h^{lm}\partial_l\phi\partial_m\phi - m^2\phi^2] \right\} : \\ =& : \partial_i\phi\partial_k\phi : + h_{ik}[: \pi^2 : - T_{00}], \end{aligned} \tag{4.9.3}$$

and for a mean value

$$(\Psi, T_{ik}\Psi) = (\Psi, : \partial_i\phi\partial_k\phi : \Psi) + h_{ik}(\Psi, [: \pi^2 : - T_{00}]\Psi). \tag{4.9.4}$$

5 Applications to cosmology

5.1 The metric-consistent energy-momentum tensor

Let in eq.(4.9.4)

$$(\Psi, : \partial_i\phi\partial_k\phi : \Psi) \propto h_{ik} \tag{5.1.1}$$

hold, i.e.,

$$(\Psi, : \partial_i\phi\partial_k\phi : \Psi) = C h_{ik} h^{lm}(\Psi, : \partial_l\phi\partial_m\phi : \Psi). \tag{5.1.2}$$

Since

$$h^{ik}h_{ik} = 3, \tag{5.1.3}$$

we find

$$C = \frac{1}{3} \tag{5.1.4}$$

and by eqs.(4.9.4),(4.9.1)

$$(\Psi, T_{ik}\Psi) = \frac{1}{3} h_{ik}(\Psi, \{2 : \pi^2 : - T_{00} - m^2 : \phi^2 :\}\Psi). \tag{5.1.5}$$

5.2 A homogeneous state

Let Ψ be a homogeneous state, so that

$$\begin{aligned} &(\Psi, \{2 : \pi^2 : - T_{00} - m^2 : \phi^2 :\}\Psi) = \tfrac{1}{V} \textstyle\int_S dx \sqrt{|h|} (\Psi, \{2 : \pi^2 : - T_{00} - m^2 : \phi^2 :\}\Psi), \\ &\quad V = V_t = \textstyle\int_S dx \sqrt{|h(t)|}. \end{aligned} \tag{5.2.1}$$

We have

$$\int_S dx \sqrt{|h|} T_{00} = \sum_j \omega_j N_j, \tag{5.2.2}$$

$$\int_S dx\sqrt{|h|} : \pi^2 := \sum_j \omega_j N_j + \{aa + a^\dagger a^\dagger\}, \tag{5.2.3}$$

$$\int_S dx\sqrt{|h|} : \phi^2 := \sum_j \frac{1}{\omega_j} N_j + \{aa + a^\dagger a^\dagger\}. \tag{5.2.4}$$

Let

$$N_j \Psi = n_j \Psi \qquad \text{for all } j, \tag{5.2.5}$$

then

$$(\Psi, T_{ik}\Psi) = h_{ik}\frac{1}{3V}\sum_j \frac{\omega_j^2 - m^2}{\omega_j} n_j. \tag{5.2.6}$$

Thus the pressure is

$$p = \frac{1}{3V}\sum_j \frac{\omega_j^2 - m^2}{\omega_j} n_j = \frac{1}{3V}\sum_j \frac{k_j^2}{\omega_j} n_j, \tag{5.2.7}$$

whereas the energy density is

$$\rho = \frac{E}{V} = \frac{1}{V}\sum_j \omega_j n_j. \tag{5.2.8}$$

5.3 The Robertson-Walker spacetime

For the Robertson-Walker spacetime, the metric is of the form

$$h(s,t) = R^2(t)\kappa(s), \qquad \text{or} \quad h_{ik} = R^2(t)\kappa_{ik}, \tag{5.3.1}$$

so that we have

$$|h| = |\kappa| R^6, \quad |\kappa| = \det(\kappa_{ik}), \quad \sqrt{|h|} = R^3\sqrt{|\kappa|}, \quad h^{ik} = \frac{\kappa^{ik}}{R^2}, \tag{5.3.2}$$

and

$$\triangle = \frac{1}{R^2}\triangle_\kappa, \qquad \triangle_\kappa \varphi = \frac{1}{\sqrt{|\kappa|}}\partial_i\left[\sqrt{|\kappa|}\kappa^{ik}\partial_k\varphi\right]. \tag{5.3.3}$$

The equation (4.2.3) results in

$$\frac{1}{R^2(t)}\triangle_\kappa u_j = -k_j^2 u_j, \tag{5.3.4}$$

so that, in view of eq.(4.2.6),

$$\triangle_\kappa u_j = -\mu_j^2 u_j, \quad \mu_j^2 = \text{const}, \quad k_j^2(t) = \frac{\mu_j^2}{R^2(t)}, \quad u_j(s,t) = \frac{1}{R^{3/2}(t)} u_j^0(s), \tag{5.3.5}$$

and

$$\omega_j = \left[m^2 + \frac{\mu_j^2}{R^2(t)}\right]^{1/2}, \tag{5.3.6}$$

the last relation being a familiar result of cosmology.

In eq.(4.7.4) we obtain

$$u = \frac{u^0(s)}{R^{3/2}(t)}, \quad k = \frac{\mu}{R(t)}, \tag{5.3.7}$$

so that

$$|b| = \frac{3}{2\omega}\frac{dR/dt}{R} - \frac{1}{2}\frac{(\mu/R)^2}{\omega^3}\frac{dR/dt}{R} = \frac{3H}{2\omega} - \frac{1}{2}\frac{k^2}{\omega^3}H = \left(3 - \frac{k^2}{\omega^2}\right)\frac{H}{2\omega} < \frac{3H}{2\omega}, \tag{5.3.8}$$

where H is the Hubble constant.

$$\text{For} \quad H \approx \frac{1}{3}10^{-17}c^{-1} \quad \text{and} \quad \omega \sim 10^{15}c^{-1}, \qquad |b| < 10^{-32}. \tag{5.3.9}$$

With eqs.(5.2.7),(5.2.8) in mind, we have

$$k_j^2 = \frac{\nu_j^2}{V^{2/3}}, \quad \nu_j^2 = \text{const}, \qquad \omega_j = \left(m^2 + \frac{\nu_j^2}{V^{2/3}}\right)^{1/2}, \tag{5.3.10}$$

so that we find

$$\frac{dE}{dV} = \frac{d(\rho V)}{dV} = \sum_j n_j \frac{d\omega_j}{dV} = -\frac{1}{3V}\sum_j n_j \frac{k_j^2}{\omega_j} = -p, \tag{5.3.11}$$

i.e.,

$$dE = -pdV, \tag{5.3.12}$$

which is a standard relation.

5.4 Universe dynamics

In this and the next subsections, we follow the papers [7,8].

The S-projected Einstein equation yields

$$G_{ik} = 8\pi\kappa_g(\Psi, T_{ik}\Psi) \quad \Rightarrow \quad 2\ddot{R}R + \dot{R}^2 + 1 = -8\pi\kappa_g pR^2 \tag{5.4.1}$$

where κ_g is the gravitational constant; eq.(5.3.12) amounts to

$$\frac{d(\rho R^3)}{dR} = -3pR^2. \tag{5.4.2}$$

We obtain from eqs.(5.4.1),(5.4.2)

$$\frac{d}{dR}\left(R\dot{R}^2 + R - \frac{8\pi\kappa_g}{3}\rho R^3\right) = 0, \tag{5.4.3}$$

whence

$$R\dot{R}^2 + R - \frac{8\pi\kappa_g}{3}\rho R^3 = L = \text{const}. \tag{5.4.4}$$

The length L, which is an integral of motion, is called cosmic length. In accordance with this, the model considered is called the cosmic length universe.

The Friedmann universe corresponds to a particular value of the cosmic length,

$$L_{\text{Friedmann}} = 0. \tag{5.4.5}$$

In this sense, the Friedmann universe is the zero-length universe.

The value $L = 0$ results from the equation

$$G_{0\mu} = 8\pi\kappa_g(\Psi, T_{0\mu}\Psi), \tag{5.4.6}$$

which is violated by quantum jumps inherent in the generic case of interacting quantum fields.

5.5 Lifting the problem of the missing dark matter

The most important problem facing modern cosmology is that of the missing dark matter [9]. Most of the mass of galaxies and an even larger fraction of the mass of clusters of galaxies is dark. The problem is that even more dark matter is required to account for the rate of expansion of the universe.

More specifically, for the Friedmann universe, the equation

$$\Omega_0 = 2q_0 \tag{5.5.1}$$

holds, where Ω is the density parameter,

$$\Omega = \frac{\rho}{\rho_c}, \tag{5.5.2}$$

ρ_c is the critical value of ρ, q is the deceleration parameter,

$$q = -\frac{\ddot{R}R}{\dot{R}^2}, \tag{5.5.3}$$

and subscript 0 indicates present-day values. In particular, if $q_0 > 1/2$, the universe is closed and $\rho_0 > \rho_c$. But observational data give $\Omega_0 < 2q_0$. Eq.(5.5.1) reduces to

$$\Omega_0 = 1 + \frac{1}{R_0^2 H_0^2}. \tag{5.5.4}$$

From eq.(5.4.4) we obtain

$$\Omega_0 = 1 + \frac{1 - L/R_0}{R_0^2 H_0^2} \tag{5.5.5}$$

in place of eq.(5.5.4). For

$$p_0 \ll \frac{1}{3}\rho_0, \tag{5.5.6}$$

which is fulfilled, eq.(5.5.5) reduces to

$$\Omega_0 = 2q_0 - \frac{L/R_0}{R_0^2 H_0^2} \tag{5.5.7}$$

in place of eq.(5.5.1).

Eq.(5.5.7) lifts the problem.

6 An application to black holes

6.1 The Lemaître metric

In the case of a black hole, the metric in the comoving reference frame is the Lemaître metric:

$$h = \frac{1}{[(3/2r_s)(R-t)]^{2/3}} dR^2 + r_s^{2/3} \left[\frac{3}{2}(R-t)\right]^{4/3} (d\theta^2 + \sin^2\theta d\varphi^2), \tag{6.1.1}$$

where r_s is the Schwarzschild radius. The Schwarzschild coordinate is

$$r = \left(\frac{3}{2}\right)^{2/3} r_s^{1/3} (R-t)^{2/3}. \tag{6.1.2}$$

6.2 Quantum field in the comoving reference frame

With the equation (4.2.3) in mind, we find

$$\Delta\chi \equiv \Delta_{\vec{R}}\chi = \frac{1}{r^2}\partial_r[r^2\partial_r\chi] + \frac{1}{r^2\sin\theta}\partial_\theta[\sin\theta\partial_\theta\chi] + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2\chi = \Delta_{\vec{r}}\chi \tag{6.2.1}$$

where

$$\vec{R} = (R, \theta, \varphi), \qquad \vec{r} = (r, \theta, \varphi). \tag{6.2.2}$$

Thus eq.(4.2.3) reduces to

$$\Delta_{\vec{r}} u_j = -k_j^2 u_j, \tag{6.2.3}$$

whence

$$u_j = u_j(r, \theta, \varphi) \tag{6.2.4}$$

with r given by eq.(6.1.2), and

$$\frac{dk_j^2}{dt} = 0, \quad \omega_j = \left[m^2 + k_j^2\right]^{1/2}, \quad \frac{d\omega_j}{dt} = 0. \tag{6.2.5}$$

So in the comoving reference frame

$$\omega_j = \text{const}, \qquad H = \sum_j \omega_j a_j^\dagger a_j, \quad \frac{dH}{dt} = 0. \tag{6.2.6}$$

In eq.(4.7.2) we have

$$\frac{d\omega}{dt} = 0, \tag{6.2.7}$$

so that

$$b = \frac{\partial u}{\partial t} \Big/ u\omega. \tag{6.2.8}$$

We find from eq.(6.1.2)

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\left(\frac{r_s}{r}\right)^{1/2}. \tag{6.2.9}$$

By [10], in view of $\sqrt{|h|} \sim r^{3/2}$,

$$\left|\frac{\partial u}{\partial r}\right| \sim \left(k^2 + \frac{1}{r^2}\right)^{1/2} |u|, \tag{6.2.10}$$

so that

$$|b| \sim \left[\frac{k^2 + 1/r^2}{\omega^2} \frac{r_s}{r}\right]^{1/2} = \left[\frac{\omega^2 - m^2 + 1/r^2}{\omega^2} \frac{r_s}{r}\right]^{1/2}. \tag{6.2.11}$$

In particular,

$$\text{for } r \gg \lambda = \frac{2\pi}{k}, \qquad |b| \sim \left[\frac{\omega^2 - m^2}{\omega^2} \frac{r_s}{r}\right]^{1/2}. \tag{6.2.12}$$

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